Solution of the linearised Vlasov equation for collisionless plasmas evolving in external fields of arbitrary spatial and time dependence. I

1990 J. Phys. A: Math. Gen. 23 2439

(http://iopscience.iop.org/0305-4470/23/12/024)

J. Phys. A: Math. Gen. 23 (1990) 2439-2461. Printed in the UK

Solution of the linearised Vlasov equation for collisionless plasmas evolving in external fields of arbitrary spatial and time dependence I

V Škarka†§ and P V Coveney‡

† International Centre for Theoretical Physics, Trieste, Italy

‡ Department of Chemistry, University of Wales, Bangor, Gwynedd LL57 2UW, UK

Received 22 December 1989

Abstract. We solve perturbatively the linearised Vlasov equation describing inhomogeneous collisionless plasmas evolving in time-dependent external fields. The method employs an explicitly time-dependent formalism and is facilitated by the use of diagrammatic techniques. It leads to a straightforward algorithm for computing the contribution to the solution, order by order, in the external field. In the present paper we provide the solution to first order; higher orders are described in the following paper.

1. Introduction

The study of systems evolving in time-dependent external fields is of major importance in statistical mechanics; it is, moreover, of obvious significance in the context of plasma physics and controlled fusion.

A powerful method for the derivation of kinetic equations in non-equilibrium statistical mechanics was proposed by Prigogine *et al* (1969). The idea is to divide the time evolution of the distribution function exactly into two independent 'subdynamics', one providing the long-time or asymptotic kinetic behaviour of the system, the other describing the transient behaviour due to the initial preparation of the system (Balescu 1975). George (1973) extended this approach by showing that it is possible to decompose the evolution of a large homogeneous system into a *complete* set of independent subdynamics, rather than only into two parts; Škarka and George (1983) later made the generalisation to include inhomogeneous systems. The theory has been used to study irreversible processes at the microscopic level (Coveney 1988).

Recently, an explicitly time-dependent formalism has been constructed as a basis for the complete subdynamics decomposition of both isolated systems and those open to external influence (Coveney and George 1987, 1988, Coveney 1986, 1987a, b, Škarka and Coveney 1988).

It is natural to consider the application of this microscopic theory to plasmas evolving in inhomogeneous and time-dependent electric and/or magnetic fields. Previously, Balescu and Misguich (1974a, b, 1975a, b) used the basic division into two subdynamics to derive a general kinetic equation for plasmas evolving in time-dependent external fields. This equation contains, as special cases, the Landau and Vlasov equations, as well as the generalisation of the Balescu–Lenard equation; for strongly

§ Permanent address: Institute of Physics, PO Box 57, 11001 Beograd, Yugoslavia.

turbulent plasmas, it leads to the quasilinear approximations of Dupree (1966), Weinstock (1969) and Kraichnan (1972).

The Vlasov equation in the absence of external fields is itself of considerable importance within plasma physics and elsewhere. It provides a very good description of the behaviour of a plasma when collisions can be neglected, and has been the subject of intensive study over many years (for example, Backus 1960, Boutros-Ghali and Dupree 1981, Davidson 1972, Ghizzo *et al* 1988, Krall and Trivelpiece 1973, Misguich and Balescu 1982). The classic earlier work focused largely on the simpler linearised Vlasov equation, which was treated as an eigenvalue problem by van Kampen (1955, 1957) and later revamped by Case (1959). An alternative method of solution was given by Balescu (1963), based on a resolvent formalism originally introduced by Résibois (see, for example, Résibois 1967).

More recently, Škarka and George (1984) employed a complete correlation subdynamics decomposition within the resolvent formalism to obtain formal analytical solutions of the nonlinear Vlasov equation in the absence of external fields; these formal solutions have been explicitly computed by Škarka (1989). For the linear case, their treatment corresponds to that of van Kampen and Case, and so may be regarded as its generalisation.

By employing the aforementioned time-dependent formalism, one can now begin to consider the solution of the Vlasov equation in the presence of external fields. In the present paper, we use analytical techniques based on subdynamics to solve the linearised Vlasov equation in an arbitrary time-dependent external field. We emphasise that no special assumptions are made about this field. However, the solutions are based on a perturbation expansion with respect to the external field, and are therefore restricted to situations where such an expansion is valid. Our results also represent an important step towards an analogous treatment of the nonlinear Vlasov equation (which we hope to deal with in a subsequent publication). We note in passing that Mahajan (1988) has reported some interesting exact and almost exact solutions to the 'Vlasov-Maxwell' system describing a variety of plasma configurations with density, temperature and current gradients, using an approach which is quite different from our own.

It should be stressed that the subdynamics theory employed herein is rather general in scope: it was not constructed in order to handle specific problems. This fact renders its applicability to concrete situations somewhat involved. However, precisely by virtue of its generality, the theory is capable of transcending the limitations of more specialised approaches in dealing with complex problems analytically. Thus, our treatment of the Vlasov equation is also instructive as an illustration of the subdynamics formalism in action.

The paper is organised as follows. In section 2, we recall the salient concepts of the subdynamics theory at the level of the abstract operators. In section 3, we fix our basic ideas with respect to collisionless Vlasov plasmas in the absence of external fields, using the time-dependent description coupled with a diagrammatic approach originally introduced by Balescu (1963). Section 4 describes the new situation in the presence of an inhomogeneous time-dependent external field, for which the diagrammatic approach was recently developed by Škarka and Coveney (1988); in this section, the solution of the Vlasov equation is given to first order in the external field (but to all orders with respect to the internal interactions). The paper ends with some conclusions in section 5. In the appendix, we show explicitly that our solutions indeed satisfy the linearised Vlasov equation in the presence of an external field.

In order to avoid overburdening the present paper, we continue the analysis to include higher orders with respect to the external field in the following paper (Škarka and Coveney 1990).

2. Concepts from subdynamics

Our starting point is the Liouville equation for the phase-space distribution function $\rho(t)=\rho(\boldsymbol{q}_1, \boldsymbol{v}_1, \ldots, \boldsymbol{q}_N, \boldsymbol{v}_N; t)$ of a single-component plasma consisting of N structureless particles (each characterised by a mass m and charge e) which, in the absence of external fields, satisfies the Liouville equation

$$\mathrm{i}\frac{\partial}{\partial t}\rho(t)=L\rho(t). \tag{2.1}$$

L is the Liouville operator

$$L=L_0+\lambda\delta L \tag{2.2}$$

which we divide into a part corresponding to the free motion of the particles,

$$L_0=-\mathrm{i}\sum_j \boldsymbol{v}_j\cdot\boldsymbol{\nabla}_j \tag{2.3}$$

where

$$\boldsymbol{\nabla}_j=\partial/\partial\boldsymbol{q}_j \tag{2.4}$$

and a part describing the interactions between them, assumed to be mediated by a pairwise, central potential V_{jk}

$$\delta L=\mathrm{i}\sum_{j<n}\sum(\boldsymbol{\nabla}_j V_{jn})\cdot\left(\frac{\partial}{\partial\boldsymbol{v}_j}-\frac{\partial}{\partial\boldsymbol{v}_n}\right)\equiv\mathrm{i}\delta\mathscr{L} \tag{2.5}$$

λ being a coupling constant, and the summations being over each of the N particles comprising the system (indicated by subscripts j and n).

The formal solution to (2.1) can be written

$$\rho(t)=U(t)\rho(t_0)=\sum_{n=0}(U^0\lambda\delta\mathscr{L}*)^n U^0\rho(t_0) \tag{2.6}$$

where we have defined the evolution operators $U(t)=\mathrm{e}^{-\mathrm{i}Lt}$ and $U^0(t)=\mathrm{e}^{-\mathrm{i}L_0t}$; the $*$ in the second equality denotes the convolution product.

For systems evolving in the presence of time-dependent external fields, the Liouville equation becomes

$$\mathrm{i}\frac{\partial}{\partial t}\rho(t)=L^F(t)\rho(t) \tag{2.7}$$

where the total Liouville operator has acquired an explicit time-dependence

$$L^F(t)=L+\zeta\delta L^F(t) \tag{2.8}$$

with L as in (2.1) while the second term on the right-hand side describes the interaction with the external field

$$\delta L^F(t) = \sum_{j=1}^{N} \delta L_j^F(t) = \frac{1}{m} \sum_{j=1}^{N} \boldsymbol{F}_j(\boldsymbol{q}_j, \boldsymbol{v}_j; t) \cdot \frac{\partial}{\partial \boldsymbol{v}_j} \equiv \mathrm{i}\delta \mathscr{L}^F(t) \tag{2.9}$$

ξ being another coupling constant. The latter term has the form of a sum of single-particle operators because the field acts on each particle individually. Throughout this paper, we shall not seek to specify the field and the forces $\boldsymbol{F}_j$ on each particle further. From the point of view of future applications of the theory in plasma physics, however, the applied field may be any one or combination of electric and/or magnetic fields (such as the Lorentz force if laser–plasma interactions are of interest).

The formal solution of (2.7) can be written as

$$\rho(t) = U^F(t, t_0)\rho(t_0) = \sum_{n=0}^{\infty} (U\zeta\delta\mathscr{L}^F *)^n U\rho(t_0) \tag{2.10}$$

where we have further defined $U^F(t, t_0) = T\exp\{-\mathrm{i}\int_{t_0}^{t} L^F(t')\,\mathrm{d}t'\}$ (T denoting the time-ordering operator). Thus one has to deal with a double perturbation expansion, with respect to the field and the internal interactions (cf (2.6)).

2.1. Dynamics of correlations: the ground level

We shall use a convenient description of the time evolution of large dynamical systems which employs the so-called 'dynamics of correlations' for the full N-body Fourier-transformed distribution function and admits a very useful diagrammatic representation (Prigogine 1962, Balescu 1963). There is a one-to-one correspondence between each algebraic term in the perturbation series and the associated diagrams. The final aim is to compute from ρ the reduced distribution functions f_s ($1 \leqslant s \leqslant N$) which, for small values of s, determine all quantities of macoscopic interest. Since we shall be interested in taking the thermodynamic limit (or T-limit, in which the number N of particles present and the volume Ω containing them both tend to infinity in such a way that the density remains finite), the reduced distribution functions for a finite number of particles should remain finite in this limit (Balescu 1963). The T-limit is chosen as a convenient mathematical device for the computation of the properties of large systems.

At this stage, one can define a complete set of Hermitian projection operators $\{P^{(\nu)}\}$ which project out of the Fourier-transformed distribution function the νth correlated components ρ_ν. These projectors commute with the evolution operator for free motion U^0 and also with L_0.

2.2. Subdynamics for isolated systems: the first level

In the T-limit, one can define another complete set of non-Hermitian projection operators $\{\boldsymbol{\Pi}^{(\nu)}\}$ in the presence of interactions ($\lambda\delta L$); these decompose the space of the distribution function into a set of subspaces which are invariant under the motion generated by L (George 1973, Škarka and George 1983, Coveney 1987a). One can now show that the new projectors commute with U and L. Each component

$$\rho^{(\nu)}(t) \equiv \boldsymbol{\Pi}^{(\nu)}\rho(t) \tag{2.11}$$

evolves independently according to the Liouville equation and thus represents an independent subdynamics (the 'first level'). Moreover, each subdynamics has a leading or privileged component $\rho_{\nu}^{(\nu)}$ in terms of which all other components $\rho_{\mu}^{(\nu)}$ $(\mu \neq \nu)$ are given:

$$\begin{aligned}\rho_{\mu}^{(\nu)}(t) &= \mathbb{C}_{\mu\nu}^{(\nu)}\mathbb{E}^{(\nu)}(t-t_0)\mathbb{D}_{\nu\eta}^{(\nu)}\rho_{\eta}(t_0)\\ &\equiv \mathbb{C}_{\mu\nu}^{(\nu)}\mathbb{E}^{(\nu)}(t-t_0)\rho_{\nu}^{(\nu)}(t_0) \equiv \mathbb{C}_{\mu\nu}^{(\nu)}\rho_{\nu}^{(\nu)}(t).\end{aligned} \tag{2.12}$$

Here $\mathbb{C}^{(\nu)}$, and $\mathbb{D}^{(\nu)}$ are, respectively, the creation and destruction superoperators; $\mathbb{E}^{(\nu)}$ represents the evolution superoperator for the νth correlation (George (1973), Coveney (1987a)—the latter is of particular relevance as it is concerned with an explicitly time-dependent approach).

2.3. *Subdynamics for open systems: the second level or generalised subdynamics*

From the Liouville equation (2.7) and the associated evolution operator U^F it is also possible, in the T-limit, to extend the $\{\Pi^{(\nu)}\}$ to a further complete set of projectors $\{\mathbb{P}^{(\nu)}(t)\}$ (Coveney 1987b); these satisfy an intertwining relation with U^F:

$$\mathbb{P}^{(\nu)}(t)U^F(t, t_0) = U^F(t, t_0)\mathbb{P}^{(\nu)}(t_0). \tag{2.13}$$

The projections

$${}^{\nu}\rho(t) \equiv \mathbb{P}^{(\nu)}(t)\rho(t) \tag{2.14}$$

thus also obey the Liouville equation (2.7). Hence they constitute a generalised ('second level' of) subdynamics. Again, there are leading components ${}^{\nu}\rho^{(\nu)}(t) = \Pi^{(\nu)}\mathbb{P}^{(\nu)}(t)\rho$ in terms of which all others may be expressed, namely

$$\begin{aligned}{}^{\nu}\rho^{(\mu)}(t) &= \mathbb{C}_{F;\mu\nu}^{(\nu)}(t)\mathbb{E}_F^{(\nu)}(t, t_0)\mathbb{D}_{F;\nu\eta}^{(\nu)}(t_0)\rho^{(\eta)}(t_0)\\ &\equiv \mathbb{C}_{F;\mu\nu}^{(\nu)}(t)\mathbb{E}_F^{(\nu)}(t, t_0)\,{}^{\nu}\rho^{(\nu)}(t_0) \equiv \mathbb{C}_{F;\mu\nu}^{(\nu)}(t)\,{}^{\nu}\rho^{(\nu)}(t)\end{aligned} \tag{2.15}$$

$\mathbb{C}_F^{(\nu)}$ and $\mathbb{D}_F^{(\nu)}$ are respectively the field-particle creation and destruction superoperators (Coveney 1987b); $\mathbb{E}_F^{(\nu)}$ is the evolution superoperator for the νth subdynamics.

All these results are algebraically exact in the T-limit (presupposing the validity of the statistical mechanical perturbation theory). However, both levels of subdynamics require as prerequisites certain additional elements. One is a general theorem in the dynamics of correlations (Škarka 1978a, b; Škarka and Coveney 1988). Another is a well defined regularisation procedure to handle the T-limit. Such a regularisation has been provided by Coveney and George (1987) in the explicitly time-dependent formalism. It plays a key role throughout the present paper, and the reader is urged to study this paper should further clarification of the technique be required. Finally, it should be remembered that our approach is non-rigorous, and simply assumes the existence and convergence of the various quantities which occur. One has always to check for these properties in each case studied. However, we note in passing that Coveney and Penrose (1989) have very recently established some rigorous results concerning existence and convergence properties with respect to the basic subdynamics decomposition.

3. The linearised Vlasov equation without external field

In this section we illustrate how the time-dependent formalism may be used to solve the familiar linearised Vlasov equation subject to periodic boundary conditions, written in $\boldsymbol{K}$-space as

$$\frac{\partial}{\partial t} f_{\boldsymbol{K}}(\boldsymbol{v}_a;\, t) + \mathrm{i}\boldsymbol{K}\cdot\boldsymbol{v}_a f_{\boldsymbol{K}}(\boldsymbol{v}_a;\, t) = \mathrm{i}\omega_{\mathrm{p}}^2 \frac{1}{\boldsymbol{K}^2} \boldsymbol{K}\cdot\frac{\partial\varphi_a(\boldsymbol{v}_a;\, t)}{\partial \boldsymbol{v}_a} \int \mathrm{d}\boldsymbol{v}_j f_{\boldsymbol{K}}(\boldsymbol{v}_j;\, t) \tag{3.1}$$

with $f_{\boldsymbol{K}_a}(\boldsymbol{v}_a;\, t)$ and $\varphi_j(\boldsymbol{v}_j;\, t)$ the inhomogeneous and homogeneous one-particle reduced distribution functions respectively. ω_{p} is the plasma frequency which depends on the number density $c(\omega_{\mathrm{p}}^2 = 4\pi e^2 c/m)$. For the analysis of the time evolution in collisionless plasmas one starts from the formal solution of the Liouville equation (2.1) with $\lambda = e^2$, and considers only those contributions proportional to powers of the inverse plasma frequency (or equivalently, to powers of $e^2 c$) which are then resummed (Balescu 1963). As a corollary, the initial distribution function is necessarily factorised:

$$\rho_{\boldsymbol{K}_1,\ldots,\boldsymbol{K}_s}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_s;\, 0) = \prod_{j=1}^{s} f_{\boldsymbol{K}_j}(\boldsymbol{v}_j;\, 0). \tag{3.2}$$

Moreover, in the linearised case, only a single one-particle distribution function is retained. In the diagrammatic formulation, this in turn implies that one need only consider the so-called 'loop vertices' (figure 1).

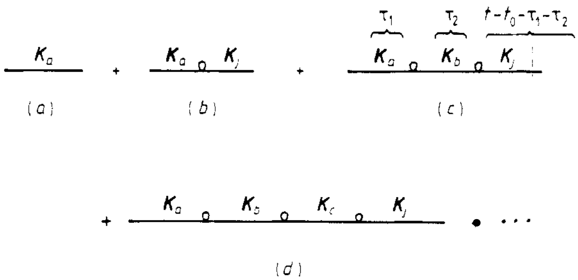

Figure 1. Summation of loop diagrams contributing to the solution of the linearised Vlasov equation (3.1).

In order to obtain the solution of the linearised Vlasov equation only fully connected diagrams of the type shown in figure 1 need to be retained (Škarka and George 1984). One proceeds by computing their contributions to each subdynamics. As an illustration of how these contributions can be written in the time-dependent formalism and then computed in the various subdynamics, consider the contribution of diagram (c) in figure 1 to the single-particle distribution function:

$$\begin{aligned} f_{\boldsymbol{K}_a}(t) \doteqdot \int_0^t \mathrm{d}\tau_1\, \mathrm{e}^{-\mathrm{i}\boldsymbol{K}\cdot\boldsymbol{v}_a\tau_1}\, \mathrm{i}\frac{8\pi^3 e^2}{m\Omega} V_{\boldsymbol{K}|}\boldsymbol{K}\cdot\frac{\partial\varphi_a}{\partial\boldsymbol{v}_a}\sum_b \int \mathrm{d}v_b \int_0^{t-\tau_1} \mathrm{d}\tau_2\, \mathrm{e}^{-\mathrm{i}\boldsymbol{K}\cdot\boldsymbol{v}_b\tau_2} \\ \times \int \mathrm{i}\frac{8\pi^3 e^2}{m\Omega} V_{|\boldsymbol{K}|}\boldsymbol{K}\cdot\frac{\partial\varphi_b}{\partial\boldsymbol{v}_b}\sum_j \int \mathrm{d}v_j\, \mathrm{e}^{-\mathrm{i}\boldsymbol{K}\cdot\boldsymbol{v}_j(t-\tau_1-\tau_2)} f_{\boldsymbol{K}_j}(0) \end{aligned} \tag{3.3}$$

(the symbol $\doteqdot$ implying that this is only one contribution to the complete solution). In what follows we prefer, in fact, to work instead with the so-called local density-excess function

$$h_{\boldsymbol{K}_a}(t) = \int \mathrm{d}\boldsymbol{v}_a f_{\boldsymbol{K}_a}(t) \tag{3.4}$$

because this is a quantity of direct physical interest (it represents the local deviation from the mean plasma density).

According to the regularisation rules established by Coveney and George (1987), we can calculate the contribution of the same diagram to the subdynamics associated with the state $\boldsymbol{K}_j$, indicated by the dotted vertical line in figure 1. Briefly, one extends to $+\infty$ the upper limits of time integrals associated with states to the left of $\boldsymbol{K}_j$ (being in this sense *more* correlated than $\boldsymbol{K}_j$), and to $-\infty$ those involving states to the right of $\boldsymbol{K}_j$ (which are *less* correlated). Note that this so-called *chronological ordering* is associated with the common convention of reading the diagrams from right to left (in the direction of increasing time). The regularisation in fact amounts to the so-called iξ-rule formulated by George (1970) in the resolvent formalism, in which iξ is added to the denominators of propagators arising from more correlated states than the one whose subdynamics is sought, while $-$iξ is added to the denominators of less correlated states. The limit $\xi \to 0^+$ is intended.

This prescription, leads to the following chain of equalities:

$$
\begin{aligned}
h_{\boldsymbol{K}_a}(t) \doteq & \int \mathrm{d}\boldsymbol{v}_a \int_0^{+\infty} \mathrm{d}\tau_1\, \mathrm{e}^{-\xi-\mathrm{i}\boldsymbol{K}\cdot(\boldsymbol{v}_a-\boldsymbol{v}_j)\tau} \frac{\mathrm{i}\omega_{\mathrm{p}}^2 \boldsymbol{K}}{K^2}\cdot\frac{\partial\varphi_a}{\partial\boldsymbol{v}_a} \int \mathrm{d}\boldsymbol{v}_b \\
& \times \int_0^{t-\tau_1} \mathrm{d}\tau_2\, \mathrm{e}^{-\xi-\mathrm{i}\boldsymbol{K}\cdot\boldsymbol{v}_b\tau_2} \frac{\mathrm{i}\omega_{\mathrm{p}}^2 \boldsymbol{K}}{K^2}\cdot\frac{\partial\varphi_b}{\partial\boldsymbol{v}_b} \int \mathrm{d}\boldsymbol{v}_j\, \mathrm{e}^{-\mathrm{i}\boldsymbol{K}\cdot\boldsymbol{v}_j t} f_{\boldsymbol{K}_j}(0) \\
= & \int \prod_n \mathrm{d}\boldsymbol{v}_n \frac{1}{\mathrm{i}\xi - \boldsymbol{K}\cdot(\boldsymbol{v}_a-\boldsymbol{v}_j)} \frac{\omega_{\mathrm{p}}^2 \boldsymbol{K}}{K^2}\cdot\frac{\partial\varphi_a}{\partial\boldsymbol{v}_a} \\
& \times \frac{1}{\mathrm{i}\xi - \boldsymbol{K}\cdot(\boldsymbol{v}_b-\boldsymbol{v}_j)} \frac{\omega_{\mathrm{p}}^2 \boldsymbol{K}}{K^2}\cdot\frac{\partial\varphi_b}{\partial\boldsymbol{v}_b} \mathrm{e}^{-\mathrm{i}\boldsymbol{K}\cdot\boldsymbol{v}_j t} f_{\boldsymbol{K}_j}(0) \\
\equiv & \int \mathrm{d}\boldsymbol{v}_j \tilde{\mathbb{C}}_{\boldsymbol{K}_a;\boldsymbol{K}_j}\, \mathrm{e}^{-\mathrm{i}\boldsymbol{K}\cdot\boldsymbol{v}_j t} \tilde{\mathbb{D}}_{\boldsymbol{K}_j;\boldsymbol{K}_j} f_{\boldsymbol{K}_j}(0)
\end{aligned}
\tag{3.5}
$$

where we have replaced V_K by its explicit form when the Coulomb potential is used. We have also performed the summations over particles; note that the summation over the particles j actually corresponds to a summation over all subdynamics because j labels the states $\boldsymbol{K}_j$ whose subdynamics is taken.

In the last line of (3.5) we have highlighted the more general structure manifested by this contribution by writing it symbolically as one term from the infinite series given in (2.12): the factor on the left of the state whose subdynamics has been evaluated corresponds to a component of the creation operator, $\tilde{\mathbb{C}}_{\boldsymbol{K}_a;\boldsymbol{K}_j}$, while that on the right represents a term in the destruction operator, $\tilde{\mathbb{D}}_{\boldsymbol{K}_j;\boldsymbol{K}_j}$. The tilde on these operators implies that we have absorbed the homogeneous distribution functions and the corresponding velocity integrals into their definitions.

In the perturbative approach, one must sum a series of such terms involving progressively higher orders in the coupling constant e^2. Due to the long range of the Coulomb interactions, each particle in a plasma interacts with a great many others. In order to describe correctly the resulting collective effects (such as screening), the perturbation series must be summed to *all* orders. Moreover, the complete single-particle distribution function corresponding to the solution of the linearised Vlasov equation (hence also $h_{\boldsymbol{K}}(t)$) is obtained as a sum over all subdynamics components, i.e., we must sum over the particle index j (Škarka and George 1984), a step already performed in (3.5). Thus we have

$$h_{\boldsymbol{K}}(t)=\int \mathrm{d}\boldsymbol{v}_j\{1+J(\boldsymbol{K})+J^2(\boldsymbol{K})+\ldots\}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{K}\cdot\boldsymbol{v}_j t}\{1+J^{\mathrm{cc}}(\boldsymbol{K})+[J^{\mathrm{cc}}(\boldsymbol{K})]^2+\ldots\}f_{\boldsymbol{K}_g}(0)$$

$$=\int \mathrm{d}\boldsymbol{v}_j\frac{1}{1-J(\boldsymbol{K})}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{K}\cdot\boldsymbol{v}_j t}\,\frac{1}{1-J^{\mathrm{cc}}(\boldsymbol{K})}f_{\boldsymbol{K}_g}(0) \tag{3.6}$$

where we have recognised that the infinite series in J are geometric and so may be summed in closed form. The common term J and its complex conjugate J^{cc} are directly related to the plasma dielectric function ε through the equation

$$\varepsilon(\boldsymbol{K})=1-J(\boldsymbol{K})=1-\frac{\omega_{\mathrm{p}}^2}{K^2}\int \mathrm{d}\boldsymbol{v}_x\frac{(-1)}{\mathrm{i}\xi-\boldsymbol{K}\cdot(\boldsymbol{v}_x-\boldsymbol{v}_y)}\boldsymbol{K}\cdot\frac{\partial\varphi_X}{\partial\boldsymbol{v}_X}. \tag{3.7}$$

Using ε, we can write the local density excess in the form

$$h_{\boldsymbol{K}}(t)=\int \mathrm{d}\boldsymbol{v}_j\frac{1}{\varepsilon(\boldsymbol{K})}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{K}\cdot\boldsymbol{v}_j t}\,\frac{1}{\varepsilon^{\mathrm{cc}}(\boldsymbol{K})}f_{\boldsymbol{K}_g}(0)$$

$$=\int \mathrm{d}\boldsymbol{v}_j\tilde{\mathbb{C}}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{K}\cdot\boldsymbol{v}_j t}\tilde{\mathbb{D}}f_{\boldsymbol{K}_g}(0)=\int \mathrm{d}\boldsymbol{v}_j\tilde{\mathbb{C}}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{K}\cdot\boldsymbol{v}_j t}f^{(\mathrm{p})}_{\boldsymbol{K}_j}(0) \tag{3.8}$$

where in the second line we have introduced the creation and destruction superoperators $\mathbb{C}$ and $\mathbb{D}$.

As an important aside which we shall later wish to employ, we note that, in line with the chain of identities in (2.12) and (3.8), we may absorb the destruction operator into the initial-time privileged component (the so-called 'post-initial' distribution function $f^{(\mathrm{p})}$). By means of this ruse, only the creation operator need be computed. As a corollary, one need only compute the subdynamics associated with the states present at the extreme right-hand side of each diagram considered. Each such diagram is then considered to represent an infinite class of diagrams having the same topological structure up to the immediate left of the correlation state whose subdynamics is computed, while differing on the right-hand side of this state (the latter part determining the destruction operator). This valuable simplification has already been utilised by Škarka and George (1984).

Škarka and George (1984) have also observed (using the resolvent formalism) that the results mentioned here coincide with expressions arising from the approaches of van Kampen (1955, 1957) and Case (1959). Indeed, each subdynamics ($\boldsymbol{K}_j$) describes free propagation of an undamped plasma mode corresponding to the eigenvalue $\boldsymbol{K}_j\cdot\boldsymbol{v}_j$; the superposition of these plane-wave eigenfunctions (summation over subdynamics) then yields the exact solution of the linearised Vlasov equation. Consequently, the extension of the subdynamics approach (based on the use of singular distributions *à la* van Kempen) to the nonlinear Vlasov equation amounts to a generalisation of the van Kampen–Case treatment (itself equivalent to Landau's Laplace transformation solution based upon the method of characteristics).

However, our main purpose in the present paper is different. Having demonstrated that the time-dependent formalism indeed reproduces the well-known results for the linearised Vlasov equation, we now turn to a consideration of the solution of the linear

Vlasov equation in the presence of time-dependent external fields of arbitrary inhomogeneity.

4. The linearised Vlasov equation in the presence of an external field

In the presence of a time-dependent external field, the linearised Vlasov equation with periodic boundary conditions assumes the $\boldsymbol{K}$-space form

$$\begin{aligned}\frac{\partial}{\partial t} f_{\boldsymbol{K}}(\boldsymbol{v}_a;\, t) + \mathrm{i}\boldsymbol{K}\cdot\boldsymbol{v}_a f_{\boldsymbol{K}}(\boldsymbol{v}_a;\, t) \\ &= \mathrm{i}\omega_{\mathrm{p}}^2 V_{|\boldsymbol{K}|}\boldsymbol{K}\cdot\frac{\partial\varphi_a}{\partial \boldsymbol{v}_a}\int \mathrm{d}\boldsymbol{v}_j f_{\boldsymbol{K}}(\boldsymbol{v}_j;\, t) \\ &\quad + \int \mathrm{d}\boldsymbol{K}_1 \mathrm{i}\boldsymbol{F}_{\boldsymbol{K}_1}(\boldsymbol{v}_a;\, t)\cdot\frac{\partial}{\partial \boldsymbol{v}_a} f_{(\boldsymbol{K}+\boldsymbol{K}_1)}(\boldsymbol{v}_a;\, t)\end{aligned} \tag{4.1}$$

where $\boldsymbol{F}_{\boldsymbol{K}}$ is the Fourier transform of the force due to the external field (equation (2.9)). The general subdynamics theory for systems evolving in time-dependent external fields was given by Coveney (1987b) (see section 2.3); the formal solution of the Liouville equation involves a double perturbation series, with respect to both internal interactions ($e^2\delta L$) and the external field ($\zeta\delta L^F$) (equation (2.10)). Commencing with this formal solution, we use the same criterion as in section 3 to choose an appropriate set of diagrams which now include, in addition to loop vertices, external field vertices which arise by virtue of the perturbation expansion with respect to the external field (the latter having been introduced by Škarka and Coveney 1988). These are shown in figure 2, where the diagrams are classified according to the number of external field vertices present, the dots between internal interaction loops indicating all possible numbers of additional loops, so that for instance the first diagram $\{M\}$ corresponds to the sum of the whole family of diagrams in figure 1. Moreover, we show in the appendix that this choice of diagrams indeed corresponds to the solution of the linearised Vlasov equation in an external field.

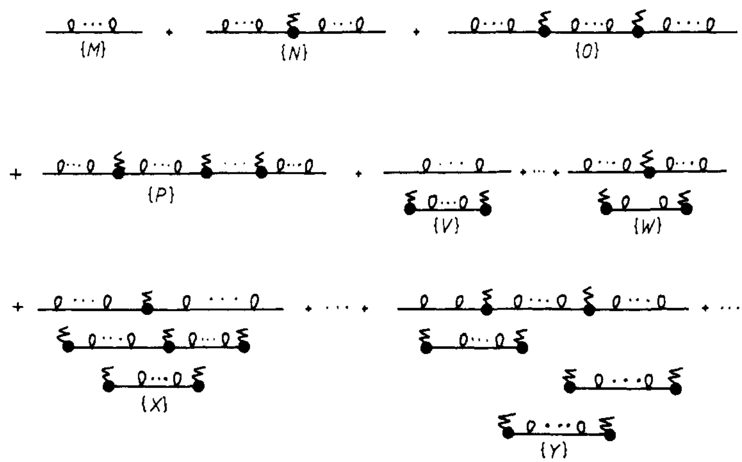

Figure 2. Summation of diagrams contributing to the solution of the linearised Vlasov equation (4.1) in the presence of an external field.

The appearance of more than one line in some of these diagrams (e.g. $\{V\}$, $\{W\}$, $\{X\}$, $\{Y\}$, etc)—the additional lines can only start and end at external field vertices (Škarka and Coveney 1988)—is a mark of the non-trivial modification of the dynamics caused by an inhomogeneous external field.

4.1. Diagrams with a single external field vertex

As before, each diagram has a one-to-one correspondence with a particular term in the (double) perturbation series. For example, consider a particular diagram from the class $\{N\}$ in figure 2; this is reproduced in greater detail in figure 3, which shows explicitly the various time variables entering into the nested convolution whose

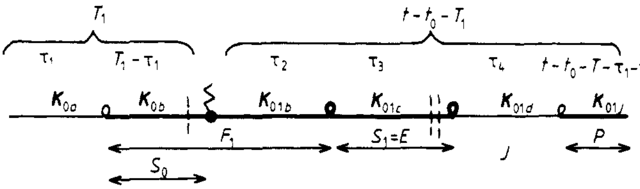

Figure 3. A single diagram in the class $\{N\}$ (in figure 2), displaying wavevectors and time intervals.

algebraic expression is written following the one-to-one correspondence

$$
\begin{aligned}
h_{K_{0a}}(t) \doteqdot & \int \mathrm{d}\boldsymbol{v}_a \int \mathrm{d}\boldsymbol{v}_e \int \mathrm{d}\boldsymbol{v}_b \int \mathrm{d}\boldsymbol{v}_d \int \mathrm{d}\boldsymbol{v}_j \int_0^{t-t_0} \mathrm{d}T \int_0^{T_1} \mathrm{d}\tau_1 \, \mathrm{e}^{-\mathrm{i}\boldsymbol{K}_0\cdot\boldsymbol{v}_a\tau_1} \mathrm{i}\,\frac{\omega_\mathrm{p}^2 \boldsymbol{K}_0}{K_0^2} \cdot \frac{\partial \varphi_a}{\partial \boldsymbol{v}_a} \\
& \times \mathrm{e}^{-\mathrm{i}\boldsymbol{K}_0\cdot\boldsymbol{v}_b(T-\tau_1)} \int \mathrm{d}\boldsymbol{K}_1 \int \mathrm{d}\omega_1 \, \mathrm{i}\boldsymbol{F}_{\boldsymbol{K}_1}(\omega_1, \boldsymbol{v}_b) \, \mathrm{e}^{-\mathrm{i}\omega_1(t-T_1)} \\
& \times \frac{\partial}{\partial \boldsymbol{v}_b} \int_0^{t-t_0-T_1} \mathrm{d}\tau_2 \, \mathrm{e}^{-\mathrm{i}(\boldsymbol{K}_0+\boldsymbol{K}_1)\cdot\boldsymbol{v}_b\tau_2} \mathrm{i}\,\frac{\omega_\mathrm{p}^2(\boldsymbol{K}_0+\boldsymbol{K}_1)}{(\boldsymbol{K}_0+\boldsymbol{K}_1)^2} \cdot \frac{\partial \varphi_b}{\partial \boldsymbol{v}_b} \int_0^{t-t_0-T_1-\tau_2} \mathrm{d}\tau_3 \\
& \times \mathrm{e}^{-\mathrm{i}(\boldsymbol{K}_0+\boldsymbol{K}_1)\cdot\boldsymbol{v}_b\tau_3} \mathrm{i}\,\frac{\omega_\mathrm{p}^2(\boldsymbol{K}_0+\boldsymbol{K}_1)}{(\boldsymbol{K}_0+\boldsymbol{K}_1)^2} \cdot \frac{\partial \varphi_c}{\partial \boldsymbol{v}_c} \int \mathrm{d}\tau_4 \\
& \times \mathrm{e}^{-\mathrm{i}(\boldsymbol{K}_0+\boldsymbol{K}_1)\cdot\boldsymbol{v}_d\tau_4} \mathrm{i}\,\frac{\omega_\mathrm{p}^2(\boldsymbol{K}_0+\boldsymbol{K}_1)}{(\boldsymbol{K}_0+\boldsymbol{K}_1)^2} \cdot \frac{\partial \varphi_d}{\partial \boldsymbol{v}_d} \\
& \times \mathrm{e}^{-\mathrm{i}(\boldsymbol{K}_0+\boldsymbol{K}_1)\cdot\boldsymbol{v}_j(t-t_0-T_1-\tau_1-\tau_2-\tau_3-\tau_4)} f_{\boldsymbol{K}_{01}}(\boldsymbol{v}_j;\, t_0).
\end{aligned}
\tag{4.2}
$$

Note that, for convenience, the external field is written in ω-space by Fourier-transforming with respect to t:

$$
F_{\boldsymbol{K}}(t, \boldsymbol{v}) = \int_{-\infty}^{\infty} F_{\boldsymbol{K}}(\omega, \boldsymbol{v}) \, \mathrm{e}^{-\mathrm{i}\omega t} \, \mathrm{d}\omega. \tag{4.3}
$$

(See also the appendix for an explicit computation involving the external field vertex alone.) Observe also (equation (4.2)) that the velocity derivatives associated with the internal interaction vertices act only on the appropriate initial homogeneous single-particle distribution functions φ_i. By contrast, the velocity derivatives associated with the external field vertices continue to act on other parts of the expression. It is therefore very convenient to introduce abbreviations for both the internal interaction vertex

$$\mathcal{V}_{0a} \equiv \frac{\omega_p^2}{\boldsymbol{K}_0^2} \boldsymbol{K}_0 \cdot \frac{\partial \varphi_a}{\partial \boldsymbol{v}_a} \tag{4.4}$$

and the external field vertex

$$\boldsymbol{F}_{1b} \equiv \boldsymbol{F}_{\boldsymbol{K}_1}(\omega_1, v_b)\, \mathrm{e}^{-\mathrm{i}\omega_1 t}. \tag{4.5}$$

We shall find it convenient to introduce at this point some additional definitions for the velocity difference, and the wavevector and frequency sums

$$\begin{aligned} \boldsymbol{v}_{ij} &= \boldsymbol{v}_i - \boldsymbol{v}_j \\ \boldsymbol{K}_{01} &= \boldsymbol{K}_0 + \boldsymbol{K}_1 \\ \omega_{12} &= \omega_1 + \omega_2 \end{aligned} \tag{4.6}$$

as well as a compact notation representing velocity-, ω- and $\boldsymbol{K}$-integrals

$$\int \mathrm{d}\Gamma \equiv \int \mathrm{d}\boldsymbol{v}_a \ldots \int \mathrm{d}\boldsymbol{v}_j \int \mathrm{d}\omega_1 \ldots \int \mathrm{d}\omega_m \int \mathrm{d}\boldsymbol{K}_1 \ldots \int \mathrm{d}\boldsymbol{K}_n. \tag{4.7}$$

Now (4.2) represents a description in the dynamics of correlations (the 'ground level' of section 2.1). Note that the external field vertex introduces a new feature: on either side of this vertex one has convolutions of the form previously discussed (section 3), but these are now coupled together by virtue of the convolution with the field. The field vertex divides the diagram into intervals of time denoted by capital letters (T_1, T_2, etc). Within each such interval the system evolves through a succession of internal interactions (loop vertices); as in section 3, these are denoted by Greek letters (τ_1, τ_2, etc).

In order to obtain the contribution of this diagram to the single-particle distribution function or, as we prefer, the local density excess (equation (3.4)), one computes initially the contributions to the first level subdynamics (section 2.2). Within each time interval T_i we can choose one state labelled by a wavevector with which a first level subdynamics may be associated. There will be as many such dynamics as there are time intervals T_i (including the interval $\{t - t_0 - \Sigma_i T_i\}$), delimited by external field vertices (see figures 2-5). Thus, evaluating separately the first level $\boldsymbol{K}_{01c}$-subdynamics to the right of the field vertex and the $\boldsymbol{K}_{0b}$-subdynamics to the left (each denoted by dotted vertical lines in figure 3) in the usual way (as described in section 3), (4.2) becomes

$$\begin{aligned} h_{\boldsymbol{K}_{0a}}(t) \doteq \int \mathrm{d}\Gamma \int_0^{+\infty} & \mathrm{d}\tau_1\, \mathrm{e}^{-(\xi + \mathrm{i}\boldsymbol{K}_0 \cdot \boldsymbol{v}_{ab})\tau_1}\, \mathrm{i}\, \mathcal{V}_{0a} \int_0^{t-t_0} \mathrm{d}T_1 \\ & \times \mathrm{e}^{\mathrm{i}(\omega_1 - \boldsymbol{K}_0 \cdot \boldsymbol{v}_b + \boldsymbol{K}_{01} \cdot \boldsymbol{v}_j) T_1}\, \mathrm{i}\boldsymbol{F}_{1b} \cdot \frac{\partial}{\partial \boldsymbol{v}_b} \int_0^{+\infty} \mathrm{d}\tau_2\, \mathrm{e}^{-(\xi + \mathrm{i}\boldsymbol{K}_{01} \cdot \boldsymbol{v}_{bj})\tau_2}\, \mathrm{i}\mathcal{V}_{1b} \\ & \times \int_0^{-\infty} \mathrm{d}\tau_4\, \mathrm{e}^{(\xi - \mathrm{i}\boldsymbol{K}_{01} \cdot \boldsymbol{v}_{dj})\tau_4}\, \mathrm{i}\mathcal{V}_{1c} \int_{+\infty}^{t - t_0 - T - \tau_2 - \tau_4} \mathrm{d}\tau_3\, \mathrm{e}^{-(\xi + \mathrm{i}\boldsymbol{K}_{01} \cdot \boldsymbol{v}_{cj})\tau_3}\, \mathrm{i}\mathcal{V}_{1d} \\ & \times \mathrm{e}^{-\mathrm{i}\boldsymbol{K}_{01} \cdot \boldsymbol{v}_j (t - t_0)} f_{\boldsymbol{K}_{01}}(\boldsymbol{v}_j;\, t_0). \end{aligned} \tag{4.8}$$

In order to integrate first over the time interval τ_3 corresponding to the chosen subdynamics, the integrals over τ_2 and τ_4 have to be interchanged following the prescription of Coveney and George (1987). However, all other integrations follow the same ordering as in figure 3. After integration over all τ-intervals we obtain

$$
\begin{aligned}
h_{K_{0a}}(t) \doteqdot & \int \mathrm{d}\Gamma \frac{1}{\mathrm{i}\xi - K_0 \cdot v_{ab}} \mathscr{V}_{0a} \int_0^{t-t_0} \mathrm{d}T_1\, \mathrm{e}^{-\mathrm{i}K_0 \cdot v_b T_1}\, \mathrm{i}F_{1b}\, \mathrm{e}^{\mathrm{i}\omega_1 T_1} \cdot \frac{\partial}{\partial v_b} \\
& \times \frac{1}{\mathrm{i}\xi - K_{01} \cdot v_{bc}} \mathscr{V}_{1b}\, \mathrm{e}^{-\mathrm{i}K_{01} \cdot v_c (t-t_0-T_1)} \mathscr{V}_{1c} \frac{1}{-\mathrm{i}\xi - K_{01} \cdot v_{dc}} \mathscr{V}_{1d} \\
& \times \frac{1}{-\mathrm{i}\xi - K_{01} \cdot v_{jc}} f_{K_{01}}(v_j;\, t_0) = \int \mathrm{d}v_b \int_0^{t-t_0} \mathrm{d}T_1 \tilde{\mathbb{C}}_b\, \mathrm{e}^{-\mathrm{i}K_0 \cdot v_b T_1} \tilde{\mathbb{D}}_b \\
& \times \mathrm{i}F_{1b}\, \mathrm{e}^{\mathrm{i}\omega_1 T_1} \cdot \frac{\partial}{\partial v_b} \int \mathrm{d}v_c \tilde{\mathbb{C}}_c\, \mathrm{e}^{-\mathrm{i}K_{01} \cdot v_c (t-t_0-T_1)} \tilde{\mathbb{D}}_c f_{K_{01}}(v_j;\, t_0)
\end{aligned}
\tag{4.9}
$$

where in the last line we have rewritten the expression in terms of creation and destruction operators appropriate to the first level subdynamics (cf (3.5)). For this particular case, however, in the interval T_1 we have only a contribution from a creation operator, because of the choice of subdynamics made ($\mathbb{D} = 1$).

We next pass to the second level of subdynamics (section 2.3), repeating the same series of manipulations but now with respect to the variables T_1, T_2, etc. In the first level we took the subdynamics associated with some given wavevector; second level subdynamics can only be taken with respect to our previously chosen first level subdynamics (i.e. K_{01c} and K_{0b} in the present case). Thus, for example, if we wish to isolate the second level K_{01c}-subdynamics from this diagram (denoted by a second dotted vertical line in figure 3), we extend the upper limit of the T_1 integral to $+\infty$, since the time-independent subdynamics K_{0b} to the left of the field vertex is more correlated:

$$
\begin{aligned}
h_{K_{0a}}(t) \doteqdot & \int \mathrm{d}v_b \left\{ \int \mathrm{d}v_a \frac{1}{\mathrm{i}\xi - K_0 \cdot v_{ab}} \mathscr{V}_{0a} \right\} \int \mathrm{d}v_c \frac{1}{\mathrm{i}\xi - K_0 \cdot v_b + K_{01} \cdot v_c + \omega_1} \\
& \times F_{1b} \cdot \frac{\partial}{\partial v_b} \frac{1}{\mathrm{i}\xi - K_{01} \cdot v_{bc}} \mathscr{V}_{1b}\, \mathrm{e}^{-\mathrm{i}K_{01} \cdot v_c (t-t_0)} \mathscr{V}_{1c} \\
& \times \left\{ \int \mathrm{d}v_d \frac{1}{-\mathrm{i}\xi - K_{01} \cdot v_{dc}} \mathscr{V}_{1d} \right\} \int \mathrm{d}v_j \frac{1}{-\mathrm{i}\xi - K_{01} \cdot v_{jc}} f_{K_{01}}(v_j;\, t_0) \\
= & \int \mathrm{d}v_c \mathbb{C}^F\, \mathrm{e}^{-\mathrm{i}K_{01} \cdot v_c (t-t_0)} \mathbb{D}^F f_{K_{01}}(v_j;\, t_0).
\end{aligned}
\tag{4.10}
$$

We emphasise that in the final expression the same ordering of propagators and interaction terms reappears as in the corresponding diagram (figure 3). In the final line, we have again formally grouped the terms involving the creation and destruction operators which are now, however, time-dependent, and between which sits the propagator $\mathbb{E}$ for the generalised subdynamics associated with the state of interest K_{01c} (cf section 2.3, equation (2.15) and Coveney 1987b). Again, because of the choice of subdynamics made, in fact we have only a (second level) creation operator in this case. In general the destruction superoperator can be included in the initial distribution function defined as the post-initial one (Škarka and George 1984). Therefore, in the same way as was done in section 3, we can always choose the generalised subdynamics to be associated with the initial first level subdynamics (on the extreme right of the expression), taking into account that the contributions from all other possible second level subdynamics are already included in the post-initial distribution functions in the lower-order expressions.

In (4.10) we recognise integrals over the same velocity variable as the one appearing in the corresponding propagator and derivative (the two terms in braces). They are on the left- and right-hand sides of their respective subdynamics propagators, and therefore they correspond to $J(\boldsymbol{K}_0)$ and $J^{cc}(\boldsymbol{K}_{01})$ (equation (3.6)) respectively. These J-terms are independent of the others because each loop vertex changes the particle label. Note that the two propagators immediately before and after the external vertex (corresponding to the part F of the diagram in figure 3 associated with the bold line), which necessarily share the same particle (b), cannot be included in these J quantities; this is also the case for the propagator corresponding to the initial state P (drawn in bold), since there is no vertex (derivative) on its right. The same is true for the propagators associated with the subdynamics which, together with their derivatives, we denote S_0 and S_1, since they are respectively the contributions from the parts S_0 and S_1 of the diagram (appearing there also in bold). Since the propagator S_1 is chosen to be associated with the generalised subdynamics, it is relabelled E as in figure 3. Here, however, S_0 partially coincides with F, and so we denote the whole sequence ${}_{s_0}F$. If $S_1 = E$ partially coincides with F, we use the notation F_E to maintain the temporal ordering of the sequences. It is important to observe that $J(\boldsymbol{K}_0)$ and $J^{cc}(\boldsymbol{K}_{01})$ were already obtained when the first level of subdynamics was computed and remain unchanged when the second level is taken. This is because the latter is realised from the convolution of exponential propagators containing wavevectors associated with first level subdynamics alone.

Using the symbols F, S_0, $E = S_1$, J, etc to denote both the algebraic terms and the corresponding parts of the diagram whence they arise, we can write (4.10) in the following way:

$$JF_{S_0}EJ^{cc}P \tag{4.11}$$

since the contributions from each bold section of the diagram are unaltered by the presence of the others.

Let us now consider the whole family of diagrams $\{N\}$ in figure 2 consisting of a single external field vertex and all possible numbers of loops, which we reproduce in more detail in figure 4.

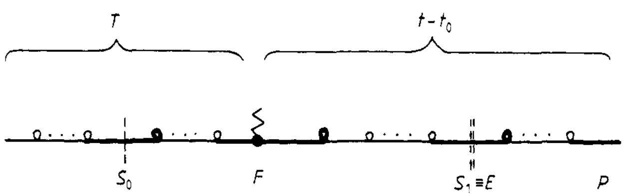

Figure 4. The class $\{N\}$ of diagrams with a choice of subdynamics made.

For progressively increasing numbers of loop vertices, their contributions correspond to the terms $J(\boldsymbol{K})$, $J^2(\boldsymbol{K})$, $J^3(\boldsymbol{K}), \ldots$, at the right of the state associated with the first level subdynamics and $J^{cc}(\boldsymbol{K})$, $\{J^{cc}(\boldsymbol{K})\}^2, \ldots$, at its left. As we described in section 3, both of these series of terms form infinite geometrical progressions whose sums are respectively $1/\varepsilon(\boldsymbol{K})$ and $1/\varepsilon^{cc}(\boldsymbol{K})$. Formally, ε belongs to the creation operator, while ε^{cc} belongs to the destruction operator. At each side of the external field vertex the same structure will appear, as illustrated in figure 4. When we calculate the second-level subdynamics these contributions, and consequently the dielectric function, remain unaltered.

We can now proceed to write down the contribution arising from the sum of the whole family of such diagrams with the same choice of subdynamics

$$h_{K_{0a}}(t) \doteqdot \int \mathrm{d}\Gamma \frac{1}{\varepsilon(\boldsymbol{K}_0)} \frac{1}{\mathrm{i}\xi - \boldsymbol{K}_0 \cdot \boldsymbol{v}_g + \boldsymbol{K}_{01} \cdot \boldsymbol{v}_c + \omega_1} \mathscr{V}_{0g} \frac{1}{\varepsilon^{\mathrm{cc}}(\boldsymbol{K})} \frac{1}{-\mathrm{i}\xi - \boldsymbol{K}_0 \cdot \boldsymbol{v}_{bg}} \boldsymbol{F}_{1b} \cdot \frac{\partial}{\partial \boldsymbol{v}_b}$$

$$\times \frac{1}{\mathrm{i}\xi - \boldsymbol{K}_{01} \cdot \boldsymbol{v}_{bc}} \mathscr{V}_{1b} \frac{1}{\varepsilon(\boldsymbol{K}_{01})} \mathrm{e}^{-\mathrm{i}\boldsymbol{K}_{01} \cdot \boldsymbol{v}_c (t-t_0)} \mathscr{V}_{1c} \frac{1}{\varepsilon^{\mathrm{cc}}(\boldsymbol{K}_{01})}$$

$$\times \frac{1}{-\mathrm{i}\xi - \boldsymbol{K}_{01} \cdot \boldsymbol{v}_{jc}} f_{\boldsymbol{K}_{01}}(\boldsymbol{v}_j;\, t_0). \tag{4.12}$$

We have summed over particles c and g: here the summation over particles is equivalent to a summation over subdynamics (a point already made in connection with (3.5)). Equation (4.12) can be written symbolically in the following way:

$$\frac{1}{\varepsilon} S_0 \frac{1}{\varepsilon^{\mathrm{cc}}} F \frac{1}{\varepsilon} E \frac{1}{\varepsilon^{\mathrm{cc}}} P \tag{4.13}$$

This formula does not include the cases when either one or the other or both subdynamics propagators S_0 and E coincide with propagators sharing the same particle with the external field vertex, i.e. which belong to F (denoted respectively ${}_{S_0}F$, F_E and ${}_{S_0}F_E$). The case when S_0 belongs to F has already been encountered in figure 3. In order to obtain the solution of the Vlasov equation we need to sum over *all* subdynamics. Adding the contribution from the sum of the whole family of diagrams with this particular choice of subdynamics (equation (4.10)) to that found from the other choices given in (4.12) we obtain

$$h_{K_{0a}}(t) \doteqdot \int \mathrm{d}\Gamma \frac{1}{\varepsilon(K_0)} \frac{1}{\mathrm{i}\xi - \boldsymbol{K}_0 \cdot \boldsymbol{v}_g + \boldsymbol{K}_{01} \cdot \boldsymbol{v}_c + \omega_1}$$

$$\times \left\{ \delta_{bg} + (1 - \delta_{bg}) \mathscr{V}_{0g} \frac{1}{\varepsilon^{\mathrm{cc}}(\boldsymbol{K}_0)} \frac{1}{-\mathrm{i}\xi - \boldsymbol{K}_0 \cdot \boldsymbol{v}_{bg}} \right\} \boldsymbol{F}_{1b} \cdot \frac{\partial}{\partial \boldsymbol{v}_b}$$

$$\times \frac{1}{\mathrm{i}\xi - \boldsymbol{K}_{01} \cdot \boldsymbol{v}_{bc}} \mathscr{V}_{1b} \frac{1}{\varepsilon(\boldsymbol{K}_{01})} \mathrm{e}^{-\mathrm{i}\boldsymbol{K}_{01} \cdot \boldsymbol{v}_c (t-t_0)} \mathscr{V}_{1c}$$

$$\times \frac{1}{\varepsilon^{\mathrm{cc}}(\boldsymbol{K}_{01})} \frac{1}{-\mathrm{i}\xi - \boldsymbol{K}_{01} \cdot \boldsymbol{v}_{jc}} f_{\boldsymbol{K}_{01}}(\boldsymbol{v}_j, t_0) \tag{4.14}$$

where δ_{ij} is the Kronecker delta.

The second, in general distinct, choice of second-level subdynamics arises when this subdynamics is associated with the propagator immediately to the right of the external field vertex, ($E \subset F$, a situation denoted symbolically as F_E). Then both vertex and propagator share the same particle index (as in figure 5) and do not commute owing to the presence of a common velocity derivative.

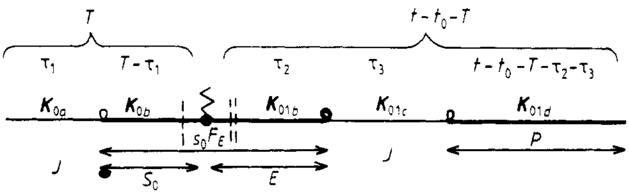

Figure 5. A diagram from the class $\{N\}$ with a particular choice of subdynamics.

In figure 5, the remaining first-level subdynamics is also associated with the state which carries the same particle index b (i.e. it is immediately to the left of the external vertex concerned). In such a situation, we denote the region around the external-field vertex by ${}_{s_0}F_E$. The contribution of this diagram is expressed as

$$\begin{aligned} h_{\boldsymbol{K}_{0a}}(t) \doteq & \int \mathrm{d}\Gamma \int_0^{+\infty} \mathrm{d}\tau_1\, \mathrm{e}^{(-\xi-\mathrm{i}\boldsymbol{K}_0\cdot\boldsymbol{v}_{ab})\tau_1}\, \mathrm{i}\,\mathcal{V}_{0a} \int_0^{t-t_0} \mathrm{d}T\, \mathrm{e}^{\mathrm{i}(\omega_1-\boldsymbol{K}_0\cdot\boldsymbol{v}_b+\boldsymbol{K}_{01}\cdot\boldsymbol{v}_d)T} \\ & \times \mathrm{i}\boldsymbol{F}_{1b}\cdot\frac{\partial}{\partial \boldsymbol{v}_b} \int_0^{-\infty} \mathrm{d}\tau_3\, \mathrm{e}^{(\xi-\mathrm{i}\boldsymbol{K}_{01}\cdot\boldsymbol{v}_{cd})\tau_3}\, \mathrm{i}\,\mathcal{V}_{1b} \int_{-\infty}^{t-t_0-T-\tau_3} \mathrm{d}\tau_2 \\ & \times \mathrm{e}^{(-\xi-\mathrm{i}\boldsymbol{K}_{01}\cdot\boldsymbol{v}_{bd})\tau_2}\, \mathrm{i}\,\mathcal{V}_{1c}\, \mathrm{e}^{-\mathrm{i}\boldsymbol{K}_{01}\cdot\boldsymbol{v}_d(t-t_0)} f_{\boldsymbol{K}_{01}}(\boldsymbol{v}_d;\, t_0). \end{aligned} \tag{4.15}$$

After integration we obtain

$$\begin{aligned} h_{\boldsymbol{K}_{0a}}(t) \doteq & \int \mathrm{d}\Gamma' \int \mathrm{d}\boldsymbol{v}_b \left\{ \int \mathrm{d}\boldsymbol{v}_a \frac{1}{\mathrm{i}\xi - \boldsymbol{K}_0\cdot\boldsymbol{v}_{ab}}\, \mathcal{V}_{0a} \right\} \boldsymbol{F}_{1b}\cdot\frac{\partial^{(1)}}{\partial \boldsymbol{v}_b} \left[\frac{\boldsymbol{K}_{01}}{\boldsymbol{K}_1}\right]^{(1)} \frac{1}{\mathrm{i}\xi + \boldsymbol{K}_1\cdot\boldsymbol{v}_b + \omega_1} \\ & \times \mathrm{e}^{-\mathrm{i}\boldsymbol{K}_{01}\cdot\boldsymbol{v}_b(t-t_0)}\, \mathcal{V}_{1b} \left\{ \int \mathrm{d}\boldsymbol{v}_c \frac{1}{-\mathrm{i}\xi - \boldsymbol{K}_{01}\cdot\boldsymbol{v}_{cb}}\, \mathcal{V}_{1c} \right\} \\ & \times \frac{1}{-\mathrm{i}\xi - \boldsymbol{K}_{01}\cdot\boldsymbol{v}_{db}} f_{\boldsymbol{K}_{01}}(\boldsymbol{v}_d;\, t_0). \end{aligned} \tag{4.16}$$

Notice that the propagator S_0, which is on the left of the vertex F in figure 5, lies to its right in (4.16). Indeed, in (4.15) the external field contains a derivative with respect to the velocity of particle b sandwiched between the subdynamical propagators S_0 and E, both of which also depend on v_b as well as on the time T. In order to integrate over T, both propagators must be collected on the same side of this derivative; as a result of their non-commutation with the derivative, a multiplicative factor (in square brackets) must be added. By definition, when a derivative carrying a superscript (1) acts on a propagator preceded by a bracket labelled by the same superscript, the propagator has to be multiplied by the expression inside the square bracket (Škarka 1989)

$$\frac{\partial^{(1)}}{\partial \boldsymbol{v}_b}\left[\frac{\boldsymbol{K}_{01}}{\boldsymbol{K}_1}\right]^{(1)} \frac{1}{\mathrm{i}\xi + \boldsymbol{K}_1\cdot\boldsymbol{v}_b + \omega_1} = \frac{-\boldsymbol{K}_1}{(\mathrm{i}\xi + \boldsymbol{K}_1\cdot\boldsymbol{v}_b + \omega_1)^2}\,\frac{\boldsymbol{K}_{01}}{\boldsymbol{K}_1} \equiv \frac{-\boldsymbol{K}_{01}}{(\mathrm{i}\xi + \boldsymbol{K}_1\cdot\boldsymbol{v}_b + \omega_1)^2}. \tag{4.17}$$

When this derivative acts on some other propagator no such multiplication occurs since there are no square brackets in front of it.

The sum of the corresponding family of diagrams for such a choice of second level subdynamics can be represented symbolically as

$$\frac{1}{\varepsilon}\, {}_{s_0}F_E \frac{1}{\varepsilon^{\mathrm{cc}}}\, P \tag{4.18}$$

Here, the subscripts preceding and following F have the meaning discussed below (4.16). Namely, when two subdynamics propagators S_0 and E share the same particles as the vertex F, the propagator S_0 appears *on the right* of the field derivative F, and is preceded by a square bracket.

Finally, let us consider the case when the first level subdynamics on the left of the external vertex is associated with a propagator not contained in F. Since the propagator corresponding to this subdynamics does not share the same particle with the external field derivative there is no bracket with a superscript. However, the term S_0 has to be placed in front of the derivative $\partial/\partial \boldsymbol{v}_b$ since it still contains the particle b originating from the second level subdynamics $(\boldsymbol{K}_{01b})$. All other terms appear in the same order as in the corresponding diagram. In symbolic notation we retain the same ordering as appears in the diagram, but with the convention that, whenever F_E occurs, the subdynamics propagator S_0 is placed to its right.

$$\frac{1}{\varepsilon} S_0 \frac{1}{\varepsilon^{\mathrm{cc}}} F_E \frac{1}{\varepsilon^{\mathrm{cc}}} P. \tag{4.19}$$

When the sum over all subdynamics of this type is taken, we obtain in compact notation

$$\begin{aligned} h_{\boldsymbol{K}_{0a}}(t) \doteqdot \int \mathrm{d}\Gamma \frac{1}{\varepsilon(\boldsymbol{K}_0)} &\left\{ \delta_{bs} + (1-\delta_{bs}) \mathcal{V}_{0s} \frac{1}{\varepsilon^{\mathrm{cc}}(\boldsymbol{K}_0)} \frac{1}{-\mathrm{i}\xi - \boldsymbol{K}_0 \cdot \boldsymbol{v}_{bs}} \right\} \boldsymbol{F}_{1b} \cdot \frac{\partial^{(1)}}{\partial \boldsymbol{v}_b} \\ &\times \left[\frac{\boldsymbol{K}_{01}}{\boldsymbol{K}_1} \right]^{(\delta_{bs})} \frac{1}{\mathrm{i}\xi - \boldsymbol{K}_0 \cdot \boldsymbol{v}_s + \boldsymbol{K}_{01} \cdot \boldsymbol{v}_b + \omega_1} \mathrm{e}^{-\mathrm{i}\boldsymbol{K}_{01} \cdot \boldsymbol{v}_b (t-t_0)} \mathcal{V}_{1b} \\ &\times \frac{1}{\varepsilon^{\mathrm{cc}}(\boldsymbol{K}_{01})} \frac{1}{-\mathrm{i}\xi - \boldsymbol{K}_{01} \cdot \boldsymbol{v}_{db}} f_{\boldsymbol{K}_{01}}(\boldsymbol{v}_d;\, t_0) \end{aligned} \tag{4.20}$$

where the bracket is activated only when its superscript is unity, i.e. when $j = s$.

We obtain the sum of all subdynamics by adding together (4.14) and (4.20) for both types of generalised subdynamics.

$$\begin{aligned} h_{\boldsymbol{K}_{0a}}(t) \doteqdot \int \mathrm{d}\Gamma \frac{1}{\varepsilon(\boldsymbol{K}_0)} &\left\{ \delta_{bg} + (1-\delta_{bg}) \mathcal{V}_{0g} \frac{1}{\varepsilon^{\mathrm{cc}}(\boldsymbol{K}_0)} \frac{1}{-\mathrm{i}\xi - \boldsymbol{K}_0 \cdot \boldsymbol{v}_{bg}} \right\} \\ &\times \boldsymbol{F}_{1b} \cdot \left\{ \delta_{bc} \frac{\partial^{(1)}}{\partial \boldsymbol{v}_b} \left[\frac{K_{01}}{K_1} \right]^{(\delta_{bg})} \frac{1}{\mathrm{i}\xi - \boldsymbol{K}_0 \cdot \boldsymbol{v}_g + \boldsymbol{K}_{01} \cdot \boldsymbol{v}_c + \omega_1} \right. \\ &\left. + (1-\delta_{bc}) \frac{1}{\mathrm{i}\xi - \boldsymbol{K}_0 \cdot \boldsymbol{v}_g + \boldsymbol{K}_{01} \cdot \boldsymbol{v}_c + \omega_1} \frac{\partial}{\partial \boldsymbol{v}_b} \frac{1}{\mathrm{i}\xi - \boldsymbol{K}_{01} \cdot \boldsymbol{v}_{bc}} \mathcal{V}_{1b} \frac{1}{\varepsilon(\boldsymbol{K}_{01})} \right\} \\ &\times \mathrm{e}^{-\mathrm{i}\boldsymbol{K}_{01} \cdot \boldsymbol{v}_c (t-t_0)} \mathcal{V}_{1c} \frac{1}{\varepsilon^{\mathrm{cc}}(\boldsymbol{K}_{01})} \frac{1}{-\mathrm{i}\xi - \boldsymbol{K}_{01} \cdot \boldsymbol{v}_{jc}} f_{\boldsymbol{K}_{01}}(\boldsymbol{v}_j;\, t_0). \end{aligned} \tag{4.21}$$

This compact expression provides the requisite sum over all subdynamics. It can be obtained in a different manner by employing the symbolic notation. This method has the advantage of a direct connection with the ordering of terms as they appear in the diagrams, and can be translated into explicit algebraic expressions using the conventions already established. Thus, we can write directly from the diagrams a symbolic expression which includes all possible choices of subdynamics:

$$\frac{1}{\varepsilon} \left\{ \left[{}_{s_0}F + S_0 \frac{1}{\varepsilon^{\mathrm{cc}}} F \right] \frac{1}{\varepsilon} E + {}_{s_0}F_E + S_0 \frac{1}{\varepsilon^{\mathrm{cc}}} F_E \right\} \frac{1}{\varepsilon^{\mathrm{cc}}} P. \tag{4.22}$$

From this representation, we can write explicit expressions for all the terms, keeping in mind the convention that each time we encounter S_0 preceding F_E, the relative ordering of these two terms must be inverted. The same is also true for the term ${}_{s_0}F_E$, with the additional insertion of a superscript (1) on the external-field derivative F and a square bracket possessing the same superscript (4.16) and (4.17)).

4.2. General formula for single external-field vertex diagrams

In order to prepare the ground for a generalisation including all single external-field vertex diagrams, the cases studied thus far in (4.2)-(4.21) will now be further analysed, taking into account the one-to-one correspondence between the diagrams and the expressions.

As we have seen, in the framework of the dynamics of correlations (ground level) the diagrams are drawn and labelled directly from the original expressions (like (4.2)) using their one-to-one correspondence (Balescu 1963). Performing the separation into first and then second level subdynamics we reach a qualitatively new stage in the description of the system. The correspondence between a final algebraic expression (such as (4.10)) and the diagram from which it originates is no longer straightforward, but can nevertheless be found (Škarka 1987). It is our task here to convince the reader of the truth of this statement when an external field is also in play.

To this end, let us return to the contributions from the diagrams considered so far. In summary, in the first or type-I choice of generalised subdynamics as the one which does not share a particle with the external field derivative ($E \not\subset F$), the terms in the final expression appear exactly in the same order as in the diagram (see the first two cases considered, equation (4.14)).

The second or type-II choice of generalised subdynamics arises when this subdynamics is associated with the propagator immediately to the right of the external field vertex so that they share the same particle ($E \subset F$) (see the last two cases considered in which F_E appears). There also, all terms in the final expression follow the same order as in the diagram, except for the term S_0 corresponding to the first level subdynamics. Although the propagator corresponding to this subdynamics appears in the diagram at the left of the external field vertex, in the final expression it is convoluted with the propagator associated with the second level subdynamics; carrying the same particle label as the derivative, this propagator must be placed to the right of it.

Each final expression (for example (4.10)) contains three kinds of propagators. The first kind are those contributing to the J, F, and P fragments in the diagram. Each such propagator is related to the corresponding propagator in the original expression (e.g. (4.2)) and therefore directly with the diagram. The original propagator contains one eigenvalue of the unperturbed Liouvillian, i.e. the wavevector $\boldsymbol{K}_x$ of the corresponding correlation multiplied by the velocity of the associated particle $\boldsymbol{v}_\alpha$. In the final propagator, from this eigenvalue is subtracted the eigenvalue $\boldsymbol{K}_x \cdot \boldsymbol{v}_\sigma$ coming from the propagator associated with the (first level) subdynamics. To this has to be added either $+\mathrm{i}\xi$ if the propagator is on the right of the subdynamical one, since it is then more correlated, or $-\mathrm{i}\xi$ if it is on the left, being then less correlated (following the '$\mathrm{i}\xi$-rule' of George (1970)).

$$\frac{1}{\pm\mathrm{i}\xi - \boldsymbol{K}_x \cdot \boldsymbol{v}_\alpha + \boldsymbol{K}_x \cdot \boldsymbol{v}_\sigma} \equiv \frac{1}{\pm\mathrm{i}\xi - \boldsymbol{K}_x \cdot \boldsymbol{v}_{\alpha\sigma}}. \tag{4.23}$$

The second kind of propagator (e.g. S_0) is that associated with the first level subdynamics. These arise when generalised subdynamics is computed, in an analogous way to the occurrence of the first kind which arise from the first level calculus. Now from the original eigenvalue $\boldsymbol{K}_x \cdot \boldsymbol{v}_\sigma$ is subtracted the eigenvalue $\boldsymbol{K}_{xy} \cdot \boldsymbol{v}_\Sigma$ corresponding to the chosen second level subdynamics. The wavevectors are different since they originate from both sides of the external field vertex. As a result of the convolution integral, in crossing this vertex ω_y has also to be added to the denominator. In addition, the sign of $\mathrm{i}\xi$ in the denominator is always positive since, having chosen the second level subdynamics to be the furthest to the right, all first level subdynamics are more correlated

$$\frac{1}{\mathrm{i}\xi - \boldsymbol{K}_x\cdot \boldsymbol{v}_\sigma + \boldsymbol{K}_{xy}\boldsymbol{v}_\Sigma + \omega_x}. \tag{4.24}$$

The third kind of propagator is the exponential corresponding to the second level subdynamics denoted by E.

The derivative always shares the particle index with the propagator to its left.

In the internal interaction term labelled by $\mathscr{V}_{y\sigma}$, the derivative acts only on the homogeneous distribution function of the same particle (σ) while the Coulomb potential depends on the current wavevector $\boldsymbol{K}_x$.

Finally, the external field term $F_{y\eta}$ does not change the particle (η) but it modifies the wavevector. It carries a dependence on the wavevector $\boldsymbol{K}_y$ corresponding to the difference of the wavevectors ($\boldsymbol{K}_x$ and $\boldsymbol{K}_{xy}$) on each side of the vertex.

The general formula for type-I expressions can be now written down in explicit form:

$$\begin{aligned}
h_{\boldsymbol{K}_{x\alpha}}(t) \doteq & \int \mathrm{d}\omega_y \int \mathrm{d}\boldsymbol{K}_y \int \mathrm{d}\boldsymbol{v}_\alpha \int \mathrm{d}\boldsymbol{v}_\sigma \frac{1}{\mathrm{i}\xi - \boldsymbol{K}_x\cdot\boldsymbol{v}_\alpha + \boldsymbol{K}_x\cdot\boldsymbol{v}_\sigma} \mathrm{i}\boldsymbol{K}_x V_{|\boldsymbol{K}_x|}\cdot\frac{\partial\varphi_\alpha}{\partial\boldsymbol{v}_\alpha}\cdots\int \mathrm{d}\boldsymbol{v}_\Sigma \\
&\times\frac{1}{\mathrm{i}\xi - \boldsymbol{K}_x\cdot\boldsymbol{v}_\sigma + \boldsymbol{K}_{xy}\cdot\boldsymbol{v}_\Sigma + \omega_y}\int \mathrm{d}\boldsymbol{v}_\beta \frac{1}{-\mathrm{i}\xi - \boldsymbol{K}_x\cdot\boldsymbol{v}_\beta + \boldsymbol{K}_x\cdot\boldsymbol{v}_\sigma} \\
&\times \mathrm{i}\boldsymbol{K}_x\cdot V_{|\boldsymbol{K}_x|}\frac{\partial\varphi_\beta}{\partial\boldsymbol{v}_\beta}\cdots\int \mathrm{d}\boldsymbol{v}_\eta\frac{1}{-\mathrm{i}\xi - \boldsymbol{K}_x\cdot\boldsymbol{v}_\eta + \boldsymbol{K}_x\cdot\boldsymbol{v}_\sigma}\mathrm{i}\boldsymbol{F}_{\boldsymbol{K}_y}(\omega_y, \boldsymbol{v}_\eta) \\
&\times \mathrm{e}^{-\mathrm{i}\omega_y t}\cdot\frac{\partial}{\partial\boldsymbol{v}_\eta}\,\frac{1}{\mathrm{i}\xi - \boldsymbol{K}_{xy}\cdot\boldsymbol{v}_\eta + \boldsymbol{K}_{xy}\cdot\boldsymbol{v}_\Sigma}\cdot\mathrm{i}\boldsymbol{K}_{xy}V_{|\boldsymbol{K}_{xy}|}\cdot\frac{\partial\varphi_\eta}{\partial\boldsymbol{v}_\eta}\int \mathrm{d}\boldsymbol{v}_\mu \\
&\times\frac{1}{\mathrm{i}\xi - \boldsymbol{K}_{xy}\cdot\boldsymbol{v}_\mu + \boldsymbol{K}_{xy}\cdot\boldsymbol{v}_\Sigma}\mathrm{i}\boldsymbol{K}_{xy}V_{|\boldsymbol{K}_{xy}|}\cdot\frac{\partial\varphi_\mu}{\partial\boldsymbol{v}_\mu}\cdots\mathrm{e}^{-\mathrm{i}\boldsymbol{K}_{xy}\cdot\boldsymbol{v}_\Sigma(t-t_0)} \\
&\times\int \mathrm{d}\boldsymbol{v}_\gamma\frac{1}{-\mathrm{i}\xi - \boldsymbol{K}_{xy}\cdot\boldsymbol{v}_\gamma + \boldsymbol{K}_{xy}\cdot\boldsymbol{v}_\Sigma}\mathrm{i}\boldsymbol{K}_{xy}V_{|\boldsymbol{K}_{xy}|}\cdot\frac{\partial\varphi_\gamma}{\partial\boldsymbol{v}_\gamma}\cdots \\
&\times\int \mathrm{d}\boldsymbol{v}_\nu\frac{1}{-\mathrm{i}\xi - \boldsymbol{K}_{xy}\cdot\boldsymbol{v}_\nu + \boldsymbol{K}_{xy}\cdot\boldsymbol{v}_\Sigma}f_{\boldsymbol{K}_{xy}}(\boldsymbol{v}_\nu;\, t_0)
\end{aligned} \tag{4.25}$$

where the dots indicate an arbitrary number of internal interaction loop vertices. This expression is rendered complete on insertion of the corresponding wavevector and particle indices taken directly from any given diagram following a straightforward algorithm; it then provides the explicit contribution of the diagram to the corresponding subdynamics (Škarka 1989). (It is understood that the particular choice of subdynamics, indicated by vertical dotted lines, is also included.)

Beginning at the *left*-hand side, this algorithm consists of writing, in the first eigenvalue of the first propagator, the wavevector and the particle read from the first propagator on the diagram. We emphasise that the algorithm constructs an expression written from the left- to the right-hand side of the diagram, in contrast to the way in which the diagrams are usually read (since time increases from left to right). The wavevector and particle indices for the second eigenvalue are read from the propagator in the diagram which is chosen to be associated with the first level subdynamics (indicated by a vertical dotted line).

Next comes the contribution from the internal interaction vertex; we have already noted that the derivative shares the same particle and wavevector index as the propagator on its left.

The wavevector index of the external field term is identical to the difference of that of the propagators to the immediate left and right of the vertex in the diagram. However, the external field derivative shares the same particle as these two propagators.

Furthermore, in the propagator of the second kind—the subdynamical one—the first eigenvalue is read from the propagator (in the diagram) associated with this first-level subdynamics, while the the second eigenvalue comes from the propagator associated with the chosen generalised subdynamics. Between these two propagators appears in general at least one external field vertex which contributes by adding to the denominator the corresponding frequency ω.

Finally, an exponential propagator is written corresponding to the state chosen to be associated with the second level subdynamics.

For the type-I choice of second level subdynamics the terms in the general formula appear in exactly the same order as in the original diagram (as in expression (4.25)). For situations of type II, the same is true except for the propagator associated with the first level subdynamics. As already remarked, this propagator does not commute with the external field derivative and must be placed on its right. In such a situation, the algebraic expression corresponding to the diagram can be obtained from the general formula (4.25) with the position of the subdynamical propagator appropriately modified.

We can now cover all four cases arising within these two choices of second level subdynamics by means of a single general expression

$$
\begin{aligned}
h_{\boldsymbol{K}_{x\alpha}}(t) \doteq \int \mathrm{d}\Gamma \frac{1}{\varepsilon(\boldsymbol{K}_x)} & \left\{ \delta_{\eta\sigma} + (1-\delta_{\eta\sigma}) \mathscr{V}_{x\sigma} \frac{1}{\varepsilon^{\mathrm{cc}}(\boldsymbol{K}_x)} \frac{1}{-\mathrm{i}\xi - \boldsymbol{K}_x \cdot \boldsymbol{v}_{\eta\sigma}} \right\} \\
& \times \boldsymbol{F}_{y\eta} \cdot \left\{ \delta_{\eta\Sigma} \frac{\partial^{(1)}}{\partial \boldsymbol{v}_\eta} \left[\frac{\boldsymbol{K}_{xy}}{\boldsymbol{K}_y} \right]^{(\delta_{\eta\sigma})} \frac{1}{\mathrm{i}\xi - \boldsymbol{K}_x \cdot \boldsymbol{v}_\sigma + \boldsymbol{K}_{xy} \cdot \boldsymbol{v}_\Sigma + \omega_y} \right. \\
& \left. + (1-\delta_{\eta\Sigma}) \frac{1}{\mathrm{i}\xi - \boldsymbol{K}_x \cdot \boldsymbol{v}_\sigma + \boldsymbol{K}_{xy} \cdot \boldsymbol{v}_\Sigma + \omega_y} \frac{\partial}{\partial \boldsymbol{v}_\eta} \frac{1}{\mathrm{i}\xi - \boldsymbol{K}_{xy} \cdot \boldsymbol{v}_{\eta\Sigma}} \mathscr{V}_{y\eta} \frac{1}{\varepsilon(\boldsymbol{K}_{xy})} \right\} \\
& \times \mathrm{e}^{-\mathrm{i}\boldsymbol{K}_{xy} \cdot \boldsymbol{v}_\Sigma (t-t_0)} \mathscr{V}_{y\Sigma} \frac{1}{\varepsilon^{\mathrm{cc}}(\boldsymbol{K}_{xy})} \frac{1}{-\mathrm{i}\xi - \boldsymbol{K}_{xy} \cdot \boldsymbol{v}_{\nu\Sigma}} f_{\boldsymbol{K}_{xy}}(\boldsymbol{v}_\nu; t_0).
\end{aligned}
\tag{4.26}
$$

This expression can be realised in concrete form with wavevector and particle indices taken from a given diagram using the above algorithm, but now taking into account that we deal directly with the sum over *all* generalised subdynamics.

This algorithm and the resulting general formula furnish us with very useful tools. They enable us to write down directly the contribution to the solution of the Vlasov equation from the sum of a family of diagrams, without the need for tedious computations.

Indeed, the general formula (4.25) and its compact version (4.26) are more far-reaching than might at first sight be concluded. For going to higher order diagrams with two or more external-field vertices does not require the computation *ab initio* of the corresponding expressions. Indeed, the very structure of the diagrams (figure 2) suggests the repetition of the single external vertex patterns which we have hitherto studied. This is underpinned by the fact that each loop vertex between external-field vertices changes the particle index of the velocity vector, making what follows independent of the velocity of the preceding particle—a property of significance since the algebraic expressions contain derivatives with respect to velocities.

At the first level the computations are done for each subdynamics independently of all others. One should recall that, in expressions (4.8) and (4.9), the contribution to the subdynamics S_0 on the left of the external interaction vertex is computed independently from that originating on the right in $S_1 = E$. As we have said before, the terms involving powers of J (equations (3.6), (3.7), (4.9) and (4.10))—important for the summation of families of diagrams—are dealt with at this stage and are not affected by computation of the generalised subdynamics. Therefore, when this second level is considered, only the subdynamics propagators participate. In the following paper (Škarka and Coveney 1990), we show how general formulae for higher-order external-field contributions follow naturally from the first-order expression given by (4.26).

5. Conclusions

We have obtained the solution of the linearised Vlasov equation describing a collisionless plasma evolving in an arbitrary spatial and time dependent external field. In the present paper, the solution was performed explicitly up to first order with respect to the external field and including all orders with respect to the internal interactions (which is necessary to handle collective effects in plasmas).

The solution corresponds to the sum of the contributions from each class of diagrams to all possible first and second level subdynamics. The general formula (4.26) for single external-field vertex diagrams enables one to obtain the solution to first order in the field by making use only of a straightforward algorithm without the need for long computations. Indeed, by using the correspondence between the diagrams and the final algebraic expressions which we have established here, the task of computing the contributions from the diagrams is reduced to the enumeration and labelling of all possible diagrams. An important role in the derivation of general formulae is played by the construction of symbolic expressions from the diagrams. The symbols can be translated into explicit algebraic expressions by means of the conventions regarding their mutual ordering established in the present paper. Such considerations are of particular interest when higher-order contributions are considered since the computations would otherwise become very time consuming. This situation is discussed in the following paper (Škarka and Coveney 1990).

In summary, our treatment offers a new, microscopic and analytically based approach to many fundamental problems in plasma physics, in particular in the domain of laser-plasma interactions where strongly inhomogeneous and time-dependent effects are of major importance. We hope to return in the future with a detailed discussion of some of these.

Acknowledgmnents

We are grateful to Professor R Balescu, Professor C George, and Dr H C Barr for helpful discussions, and to Professor I Prigogine for his encouragmeent. We are indebted to the British Council, the Serbian Ministry of Science, and the Instituts Internationaux Solvay de Physique et de Chimie (Brussels) for financial support for this project.

Appendix

In order to demonstrate that the solution we have obtained indeed corresponds to the linearised Vlasov equation in the presence of an external field (equation (4.1)), we first compute explicitly the contribution to the one-particle distribution function of the diagram with a single external-field vertex (figure 6)

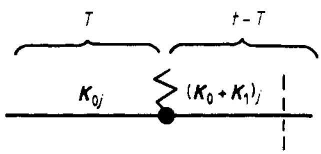

Figure 6. Diagram with a single external-field vertex.

whose algebraic form is

$$\int_0^t \mathrm{d}T\, \mathrm{e}^{-\mathrm{i}\boldsymbol{K}_0\cdot\boldsymbol{v}_j T} \int \mathrm{d}\boldsymbol{K}_1 \int \mathrm{d}\omega\, \mathrm{i}\boldsymbol{F}_{\boldsymbol{K}_1}(\omega, \boldsymbol{v}_j)\, \mathrm{e}^{-\mathrm{i}\omega(t-T)} \cdot \frac{\partial}{\partial \boldsymbol{v}_j} \mathrm{e}^{-\mathrm{i}(\boldsymbol{K}_0+\boldsymbol{K}_1)\cdot\boldsymbol{v}_j(t-T)}$$

$$= \int \mathrm{d}\boldsymbol{K}_1 \int \mathrm{d}\omega \int_0^t \mathrm{d}T\, \mathrm{e}^{-\mathrm{i}(-\boldsymbol{K}_1\cdot\boldsymbol{v}_j-\omega)T}$$

$$\times \mathrm{i}\boldsymbol{F}_{\boldsymbol{K}_1}(\omega, \boldsymbol{v}_j)\cdot\left\{\mathrm{i}(\boldsymbol{K}_0+\boldsymbol{K}_1)T + \frac{\partial}{\partial \boldsymbol{v}_j}\right\} \mathrm{e}^{-\mathrm{i}\{(\boldsymbol{K}_0+\boldsymbol{K}_1)\cdot\boldsymbol{v}_j+\omega\}t}. \tag{A1}$$

This diagram contributes the following term to the generalised $(\boldsymbol{K}_0+\boldsymbol{K}_1)_j$-subdynamics:

$$\int \mathrm{d}\boldsymbol{K}_1 \int \mathrm{d}\omega \boldsymbol{F}_{\boldsymbol{K}_1}(\omega, \boldsymbol{v}_j)\cdot\left\{\frac{(-1)}{\mathrm{i}\xi+\boldsymbol{K}_1\cdot\boldsymbol{v}_j+\omega}\frac{\partial}{\partial \boldsymbol{v}_j} + \frac{(\boldsymbol{K}_0+\boldsymbol{K}_1)}{(\mathrm{i}\xi+\boldsymbol{K}_1\cdot\boldsymbol{v}_j+\omega)^2}\right\} \mathrm{e}^{-\mathrm{i}\{(\boldsymbol{K}_0+\boldsymbol{K}_1)\cdot\boldsymbol{v}_j+\omega\}t}. \tag{A2}$$

Taking the time derivative of (A2), we find

$$\frac{\partial}{\partial t} f_{\boldsymbol{K}_0}^{(\boldsymbol{K}_{01})}(t) \doteq -\int \mathrm{d}\boldsymbol{K}_1 \int \mathrm{d}\omega \boldsymbol{F}_{\boldsymbol{K}_1}(\omega, \boldsymbol{v}_j)\cdot\left\{\frac{(-1)}{\mathrm{i}\xi+\boldsymbol{K}_1\cdot\boldsymbol{v}_j+\omega}\frac{\partial}{\partial \boldsymbol{v}_j} + \frac{(\boldsymbol{K}_0+\boldsymbol{K}_1)}{(\mathrm{i}\xi+\boldsymbol{K}_1\cdot\boldsymbol{v}_j+\omega)^2}\right\}$$

$$+\mathrm{i}\{(\boldsymbol{K}_0+\boldsymbol{K}_1)\cdot\boldsymbol{v}_j+\omega\}\,\mathrm{e}^{-\mathrm{i}\{(\boldsymbol{K}_0+\boldsymbol{K}_1)\cdot\boldsymbol{v}_j+\omega\}t} f_{\boldsymbol{K}_{01j}}^{(\boldsymbol{K}_{01})}(0). \tag{A3}$$

After separating out the 'flow term' $(\boldsymbol{K}\cdot\boldsymbol{v}_j)$, the remaining part of the exponent is combined with the propagator by invoking the Plemelj formula (Balescu 1963)

$$\frac{\partial}{\partial t} f_{\boldsymbol{K}_0}^{(\boldsymbol{K}_{01})}(t) \doteq -\mathrm{i}\boldsymbol{K}_0\cdot\boldsymbol{v}_j \int \mathrm{d}\boldsymbol{K}_1 \int \mathrm{d}\omega \boldsymbol{F}_{\boldsymbol{K}_1}(\omega, \boldsymbol{v}_j)\, \mathrm{e}^{-\mathrm{i}\omega t}$$

$$\times\left\{\frac{\boldsymbol{K}_0+\boldsymbol{K}_1}{(\mathrm{i}\xi+\boldsymbol{K}_1\cdot\boldsymbol{v}_j+\omega)^2} + \frac{(-1)}{\mathrm{i}\xi+\boldsymbol{K}_1\cdot\boldsymbol{v}_j+\omega}\frac{\partial}{\partial \boldsymbol{v}_j}\right\} f_{(\boldsymbol{K}_0+\boldsymbol{K}_1)_j}(t)$$

$$+\int \mathrm{d}\boldsymbol{K}_1 \int \mathrm{d}\omega \boldsymbol{F}_{\boldsymbol{K}}(\omega, \boldsymbol{v}_j)\cdot\left\{\mathcal{P}\left(\frac{1}{\omega+\boldsymbol{K}_1\cdot\boldsymbol{v}_j}\right) - \mathrm{i}\pi\delta(\omega+\boldsymbol{K}_1\cdot\boldsymbol{v}_j)\right\}$$

$$\times\left\{\mathrm{i}(\omega+\boldsymbol{K}_1\cdot\boldsymbol{v}_j)\frac{\partial}{\partial\boldsymbol{v}_j}+\mathrm{i}(\boldsymbol{K}_0+\boldsymbol{K}_1)-\left[\mathscr{P}\left(\frac{1}{\omega+\boldsymbol{K}_1\cdot\boldsymbol{v}_j}\right)-\mathrm{i}\pi\delta(\omega+\boldsymbol{K}_1\cdot\boldsymbol{v}_j)\right]\right.$$

$$\left.\times\,\mathrm{i}(\boldsymbol{K}_0+\boldsymbol{K}_1)(\omega+\boldsymbol{K}_1\cdot\boldsymbol{v}_j)\right\}\mathrm{e}^{-\mathrm{i}\{(\boldsymbol{K}_0+\boldsymbol{K}_1)\cdot\boldsymbol{v}_j+\omega\}t}f^{(\boldsymbol{K}_{01})}_{\boldsymbol{K}_{01j}}(0)$$

$$=\left\{-i\boldsymbol{K}_0\cdot\boldsymbol{v}_j\tilde{\mathbb{C}}^F_{\boldsymbol{K}_j}(t)+\int\mathrm{d}\boldsymbol{K}_1\int\mathrm{d}\omega\,\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\omega t}\boldsymbol{F}_{\boldsymbol{K}_1}(\omega,\boldsymbol{v}_j)\cdot\frac{\partial}{\partial\boldsymbol{v}_j}\right\}f^{(\boldsymbol{K}_{01})}_{\boldsymbol{K}_{01j}}(t). \qquad \text{(A4)}$$

($\mathscr{P}$ denotes the Cauchy principal value and δ is the Dirac δ function). Two terms are obtained, one contributing to the flow term in the Vlasov equation (4.1) and the other contributing to the external field term of the same equation.

A similar calculation can be performed in all other subdynamics for all diagrams which start at the left with an external-field vertex, following the method of Škarka and George (1984). (For diagrams of higher order with respect to the external field, we must draw on the results of the following paper (Škarka and Coveney 1990).) Then one finds that the part of each such diagram containing this vertex (labelled F in section 4) contributes precisely the expression (A4). The remainder of each diagram, i.e. the diagram with the starting vertex on the left clipped off, contributes to the time dependent, one-particle distribution function. The sum of such contributions for all possible diagrams in all subdynamics is therefore equal to the distribution function $f_{\boldsymbol{K}_{01}}(t)$; in this way we obtain the external field term in the Vlasov equation (4.1)

$$\int\mathrm{d}\boldsymbol{K}_1\int\mathrm{d}\omega\boldsymbol{F}_{\boldsymbol{K}_1}(\omega,\boldsymbol{v}_j)\,\mathrm{e}^{-\mathrm{i}\omega t}\cdot\frac{\partial}{\partial\boldsymbol{v}_j}f_{\boldsymbol{K}_{01j}}(\boldsymbol{v}_j;t). \qquad \text{(A5)}$$

Repeating the same procedure for diagrams starting with a loop vertex, we obtain the internal force term in the Vlasov equation (see Škarka and George 1984). Finally, the contributions to the flow term of both types of diagram together furnish the flow term in the Vlasov equation.

References

Backus G 1960 *J. Math. Phys.* **1** 178-94
Balescu R 1963 *Statistical Mechanics of Charged Particles* (New York: Wiley-Interscience)
—— 1975 *Equilibrium and Non-equilibrium Statistical Mechanics* (New York: Wiley-Interscience)
Balescu R and Misguich J H 1974a *J. Plasma Phys.* **11** 257-75
—— 1974b *J. Plasma Phys.* **11** 377-87
—— 1975a *J. Plasma Phys.* **13** 33-51
—— 1975b *J. Plasma Phys.* **13** 53-61
Boutros-Ghali T and Dupree T H 1981 *Phys. Fluids* **24** 1839-58
Case K 1959 *Ann. Phys., NY* **7** 349-64
Coveney P V 1986 *Acad. R. Belgique, Bull. Cl. Sci.* **72** 500-24
—— 1987a *Physica* **143A** 123-46
—— 1987b *Physica* **143A** 507-34
—— 1988 *Nature* **333** 409-15
Coveney P V and George C 1987 *Physica* **141A** 403-26
—— 1988 *J. Phys. A: Math. Gen.* **21** L869-94
Coveney P V and Penrose O 1989 unpublished
Davidson R 1972 *Methods in Nonlinear Plasma Theory* (New York: Academic)
Dupree T H 1966 *Phys. Fluids* **9** 1773-82
George C 1970 *Acad. R. Belgique, Bull. Cl. Sci.* **56** 505-19
—— 1973 *Physica* **65** 277-302
Ghizzo, A, Izrar B, Bertrand P, Fijalkow E, Feix M R and Shoucri 1988 *Phys. Fluids* **31** 72-82

Kraichnan R H 1972 *Statistical Mechanics, Proc 6th IUPAP Conf. on Statistical Mechanics* ed S A Rice, K F Freed and J C Light (Chicago, IL: University of Chicago Press) pp 201-27

Krall N A and Trivelpiece A W 1973 *Principles of Plasma Physics* (New York: McGraw-Hill)

Mahajan S M 1988 *Phys. Fluids* **B1** 43-54

Misguich J H and Balescu R 1982 *Plasma Physics* **24** 289-318

Prigogine I 1962 *Non-equilibrium statistical mechanics* (New York: Wiley-Interscience)

Prigogine I, George C and Henin F 1969 *Physica* **45** 418-34

Résibois P 1967 *Many-Particle Systems* ed E Meeron (New York: Gordon and Breach)

Škarka V 1978a, *Acad. R. Belgique, Bull. Cl. Sci.* **64** 578-96

—— 1978b *Acad. R. Belgique, Bull. Cl. Sci.* **64** 795-813

—— 1987 *Physica* **142A** 424-40

—— 1989 *Physica* **156A** 651-78

Škarka V and Coveney P V 1988 *J. Phys. A: Math. Gen.* **21** 2595-616

—— 1990 *J. Phys. A: Math. Gen.* **23** 2463-78

Škarka V and George C 1983 *Acad. R. Belgique, Bull. Cl. Sci.* **69** 210-39

—— 1984 *Physica* **127A** 473-89

van Kampen N G 1955 *Physica* **21** 949-63

—— 1957 *Physica* **63** 641-50

Weinstock J 1969 *Phys. Fluids* **12** 1045-58